Description of the lesson series
Teacher's Guide

Title	*Integers and their addition and subtraction using tokens*
Time	*4-6 school hours (depending on students' tempo and learning level)*
Grade	*Grades 5-8 (students 11-15 years old)*
Aim of the lesson cycle and its brief description	*The aim of this series of lessons is to shape the concept of negative numbers alongside addition and subtraction operations on integers using tokens.* *The scenario can be used both in earlier grades to introduce integers and in later grades for repetition.* *As students play with the concrete model (tokens), they build the concept of a negative number as the opposite of a given positive number and develop an understanding of the operation of addition as putting in tokens and subtraction as taking away tokens.* *As a result, students undertake mathematical modelling.* *This approach to shaping negative numbers is not based on the use of order in a set of integers (we do not compare a negative number to zero), it is an algebraic model that can be extended to shape the concept of an algebraic expression and its opposite, as well as the addition and subtraction of such expressions.*
Teaching materials	*Each student is given 10 round tokens of each colour (white/black) to use as tools during the lessons.*

A linguistic note on working with tokens in the context of integers and algebraic expressions:

In our scenarios, we are careful to keep the two worlds - the world of mathematics, i.e. abstractions, and the world of real objects - in our case tokens - linguistically separate. Thus, in the context of tokens, we use terms that describe their appearance: white/black round/ oblong/square token rather than the short-form white circle/rectangle/square. Similarly, in the context of tokens, we mention placing and taking away tokens – while in the context of mathematics, we discuss addition and subtraction operations. We also make a point of verbally reading action signs as add/subtract, rather than just naming them plus/minus signs. We believe that modelling arithmetic and algebraic expressions with clarity and linguistic correctness in mind is of great value and is highly recommended.

PART 2

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Part 2

Topic: Subtraction of integers using tokens

ACTIVITY 1: DICE GAME - reminder and exercise of adding whole numbers through tokens

Intended for working in pairs. If there is an odd number of students in the class, form one trio from the more gifted students (to make the pairs work as evenly as possible).

Students (alternately) throw two dice: a white one and a black one.

The number of pips on the white die represents the positive number drawn (the number of white tokens), while the number on the black die represents the negative number (the number of black tokens).

Student I's (starting) task is:

1. Throw the dice simultaneously,
2. Arrange as many corresponding tokens as indicated by the numbers on the dice,
3. Answer the question: What value does this set of tokens represent? Provide reasons for your answer,
4. Write down the addition operation and the result in your worksheet.

[Appendix: A_ENG_ Worksheet_Dice game]

The second student checks the result. If the result is correct, student I gets 1 point, if not, student II (the checker) gets a point, provided he/she can explain the error and convince the student to solve it differently.

The students then swap roles, student II performs steps 1-4, student I checks him. This is how the first round ends.

Depending on the students' calculation skills, 2-3 rounds are carried out.

The students inform the teacher of the results of the game.

INTRODUCTION TO SUBTRACTION

- What operation could describe this situation:

I have got 5 white tokens *and I take away 3 white tokens* *How many do I have left? (Students describe: 5 - 3 = 2)*
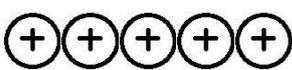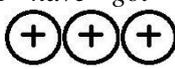
I have got 6 black tokens *and I take away one black token* . *How many do I have left? How do we write this down? (Students describe: -6 - (-1) =-5)*

AGREEMENT: TAKE AWAY MEANS SUBTRACT

ACTIVITY 2: EXERCISE OF SUBTRACTION (without adding neutral pairs)

Collaborative work. Talking.

- We are now going to deal with various subtraction situations; some can seem trivial to you ☺

- E.g. . **6 – 4** *. How could this be represented with the tokens? (*We write down the result on the board with a demonstration on the tokens, on the board).

> (Students model the situation, acivity performed on the board - taking the tokens off):
> *I have 6 white tokens, I take away 4 white tokens, so I have 2 white tokens left.*

> o *And what would an action of* **-5 – (-3)** *look like on the tokens?*
> (Students model the situation, activity performed on the board - taking the tokens off):
> *I have 5 black tokens I take away 3 black tokens, so I have 2 black tokens left*

Individual work, but summarised with student statements and notes:

- Please demonstrate the action and justify the result by using tokens

-3 – (-1) = ……

> Students arrange the model. They then provide an interpretation of the action on the model:
> *S: I have 2 black tokens and I take away 1 black token, so I am left with 1 black token.*
>
> *- Please write down the activity in your notebook and draw the justification for the result using tokens.*
> (Students cross out the subtracted token in the note. The notation appears on the board)

Individual work:

- Please represent the given operation with tokens and justify the result. Please write down the operation with the result in your notebook and draw a justification for the result with tokens.

-2 – (-2) =

-7 – (-3) =

ACTIVITY 3: PROBLEM SITUATION with the addition of neutral pairs

Activity 3 refers to the subtraction of a negative number from a positive number - in this situation we add as many neutral pairs as we have to in order to remove black tokens.

Collaborative work. Discussion:

We pose the problem to the students:
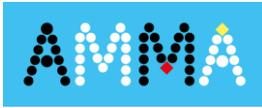
*- How do you represent the action **2 - (-1)** with tokens?* (Teacher writes the action on the board)

In turn, the teacher can say:

- I have got two white tokens. What should we do?

       S: Take one black token.

- How do we take it if we don't have it?

First, the teacher waits patiently, repeats the question. The students are given time - about a minute. If no answer is given which can be discussed*, these questions are asked in turn:

- We want to take one black token, but we need to have it first… How do we do this so that we have 1 black token but not change the value of the token set?

If no answer is given – the teacher proceeds, though slowly.

- Do you remember what the rules were for playing with tokens?

       S: White with black cancel each other out.

- Can I add something that doesn't change the value of this token set?

       S: Add a neutral pair, add one white and one black, then the value of the token set will not change.

*) Remark:

If the students state that this means adding one white token, the teacher responds that this is a very valuable point and asks them to justify it by using tokens – if the student is unable to do it, the teacher states that this idea will be revisited, so as to not disturb the current rules of the token game.

    - For practice, let's do another example together:

$2 – (-3) =$

    (Example work analogous to above, but should not require as many questions).

Individual work:

- Please represent the given operation with tokens and justify the result. Please write down the examples with the result in your notebook and draw a justification of the result with tokens for two selected activities.

$5 – (-2) =$

$3 – (-1) =$

$4 – (-6) =$

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ACTIVITY 4: Discussion of further examples with addition of neutral pairs

ACTIVITY 4A

ACTIVITY 4A concerns the subtraction of a positive number from a negative number, in this case we add as many neutral pairs as we need to remove the white tokens.

Collaborative work:

*- How do we represent the **-4 - 2** operation on tokens?*

- How many tokens do we have?

    S: We have got 4 black tokens.

- What should we do?

    S: Take away 2 white tokens.

- How do we take away 2 whites if we don't have them?

    S:...

- What should we do to have 2 more white tokens?

    S: Add two neutral pairs

Individual work:

- Please represent the given subtraction with tokens and justify the result. Please record the operation and the result in your notebook, and for two selected operations, draw the justification of the result with tokens.

    -3 – 1 =

    -2 – 3 =

    -5 – 5 =

ACTIVITY 4B

Activity 4B concerns the subtraction of negative numbers in a situation where there are not as many black tokens as there need to be taken; in this situation, adding tokens can be done in two significantly different ways.

Collaborative work. Discussion

*- How to represent the action **-2 - (-3)** with tokens?*

(When reading, we pay attention to the word SUBTRACT: minus two SUBTRACT minus three or negative two SUBTRACT negative three)

- How many tokens do we have?

    S: We have 2 black tokens.

- What should we do?

    S: Take away 3 black tokens.

- How do we do that?

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The teacher listens to the children's ideas and follows the children's spoken explanations – the emerging ideas may be described below in the **First method** or **Second method**.

Remark: If no ideas emerge, the teacher can continue asking questions such as:

- How do we take away 3 blacks when we only have 2?

- What do we do to have one more black token?

These questions steer towards the **First method**.

First method
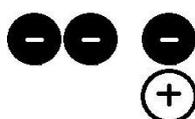
If we take away 3 black tokens now, the result is immediately : *, i.e. a value of 1.*

Another way is possible, leading to the discovery of the rule **-2 - (-3) = -2 + 3***, and we already know how to perform addition.*

Second method

We can proceed the same way as in the previous examples: in order to take away some tokens, we can first add them in the form of neutral pairs, so how would we do it in this instance? We will place 3 neutral pairs, because I have to take 3 blacks.

Then we have:
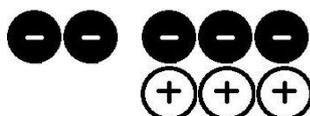
If we now take away 3 black tokens, we get the same result as **-2 + 3***:*
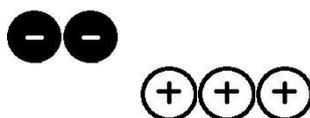
That is, **-2 - (-3) = -2 + 3.** *We already know what to do afterwards. (Although we still need to perform the second operation - addition).*

AMMA

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- To practice some more, let's do one more example together

-4 – (-6) =

(Example work analogous to the above, but should not require as many questions.)

*REMARK: We do not enforce any specific way. We analyse the methods the students are employing. If both methods appear - we analyse **both**.*

Individual work:

- Please represent the given operation with tokens and justify how the result is formed. Please write down the operation and the result in your notebook and justify it by illustrating how the result is formed by using the appropriate tokens in a drawing.

-2 – (-7) =

-3 – (-6) =

ACTIVITY 4C

Activity 4C concerns the subtraction of positive numbers in a situation where there are not as many white tokens as there are to be taken away; in this situation, adding tokens can be done in two significantly different ways.

Collaborative work. Discussion

*- How to represent the action **3 - 6** with tokens?*

(When reading, pay attention to the word SUBTRACT: 3 SUBTRACT 6)

- How many tokens do we have?

S: We have 3 white tokens.

- What should we do?

S: Take away 6 white tokens.

- How do we take away 6 white ones when we only have 3 white ones?

- What should we do if we want to have 6 white tokens?

REMARK: We implement both methods

First method

Please represent the given operation with tokens and justify how the result is formed. Please write the activity and the result in your notebook and justify by illustrating how the result is produced from the corresponding tokens in a drawing.

Reasoning:
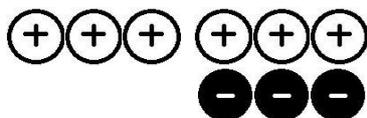
If we take away the 6 white tokens, the result is immediately : *, i.e. a value of -3.*

Second method

We can proceed like in the previous examples: in order to take away some tokens, we first placed them in the form of neutral pairs, so would this apply here? We can add 6 neutral pairs, because we are supposed to take away 6 whites.

Then we have:
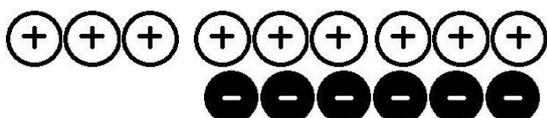
If we now take away the 6 white tokens, we get the same result as 3 + (-6):

So 3 – 6 = 3 + (-6). *We already know what to do afterwards. (Although we still need to perform the second operation - addition.)*

Individual work:

- Please represent the given operation with tokens and justify how the result is formed. Please write down the operation and the result in your notebook and justify it by using tokens to illustrate how the result is obtained.

5 – 5 =

2 – 8 =

ACTIVITY 5: SUBTRACTION – Homework

Individual work as part of a homework assignment [if there is free time until the end of the lesson, you can start during the lesson and finish at home; we will check the results of your work and analyse it].

Contents of the worksheet:

Provide the results. Highlight one action and justify it by illustrating how the result is formed from the relevant tokens in a drawing. Formulate advice on how to perform such subtractions quickly using tokens.

Task 1 $-4 - (-2) =$ $-10 - (-4) =$ $-21 - (-20) =$	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens? **Advice:**
Task 2 $3 - (-2) =$ $4 - (-5) =$ $70 - (-5) =$	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens? **advice:**
Task 3 $-1 - 3 =$ $-2 - 4 =$ $-20 - 37 =$	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens? **Advice:**
Task 4 $-1 - (-3) =$ $-2 - (-3) =$ $-10 - (-25) =$	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens? **Advice:**
Task 5 $5 - 3 =$ $7 - 10 =$ $17 - 25 =$	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens? **Advice:**

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State the results. Underline one operation and describe it by illustrating how the result is formed from the corresponding tokens in a drawing. Formulate advice on how to quickly perform such subtractions using tokens.

Task 1	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens?
$-4 - (-2) =$		**Advice:**
$-10 - (-4) =$		
$-21 - (-20) =$		
Task 2	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens?
$3 - (-2) =$		**Advice:**
$4 - (-5) =$		
$70 - (-5) =$		
Task 3	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens?
$-1 - 3 =$		**Advice:**
$-2 - 4 =$		
$-20 - 37 =$		

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State the results. Underline one operation and describe it by illustrating how the result is formed from the corresponding tokens in a drawing. Formulate advice on how to quickly perform such subtractions using tokens.

Task 4	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens?
$-1 - (-3) =$		**Advice:**
$-2 - (-3) =$		
$-10 - (-25) =$		
Task 5	Justification for the underlined operation:	How do you perform such subtractions quickly by using tokens?
$5 - 3 =$		**Advice:**
$7 - 10 =$		
$17 - 25 =$		