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Description of the lesson series
Teacher's Guide

Title	***Integers and their addition and subtraction using tokens***
Time	*4-6 school hours (depending on students' tempo and learning level)*
Grade	*Grades 5-8 (students 11-15 years old)*
Aim of the lesson cycle and its brief description	*The aim of this series of lessons is to shape the concept of negative numbers alongside addition and subtraction operations on integers using tokens.* *The scenario can be used both in earlier grades to introduce integers and in later grades for repetition.* *As students play with the concrete model (tokens), they build the concept of a negative number as the opposite of a given positive number and develop an understanding of the operation of addition as putting in tokens and subtraction as taking away tokens.* *As a result, students undertake mathematical modelling.* *This approach to shaping negative numbers is not based on the use of order in a set of integers (we do not compare a negative number to zero), it is an algebraic model that can be extended to shape the concept of an algebraic expression and its opposite, as well as the addition and subtraction of such expressions.*
Teaching materials	*Each student is given 10 round tokens of each colour (white/black) to use as tools during the lessons.*

A linguistic note on working with tokens in the context of integers and algebraic expressions:

In our scenarios, we are careful to keep the two worlds - the world of mathematics, i.e. abstractions, and the world of real objects - in our case tokens - linguistically separate. Thus, in the context of tokens, we use terms that describe their appearance: white/black round/ oblong/square token rather than the short-form white circle/rectangle/square. Similarly, in the context of tokens, we mention placing and taking away tokens – while in the context of mathematics, we discuss addition and subtraction operations. We also make a point of verbally reading action signs as add/subtract, rather than just naming them plus/minus signs. We believe that modelling arithmetic and algebraic expressions with clarity and linguistic correctness in mind is of great value and is highly recommended.

PART 1

Part 1

Topic: Integers and their addition using tokens

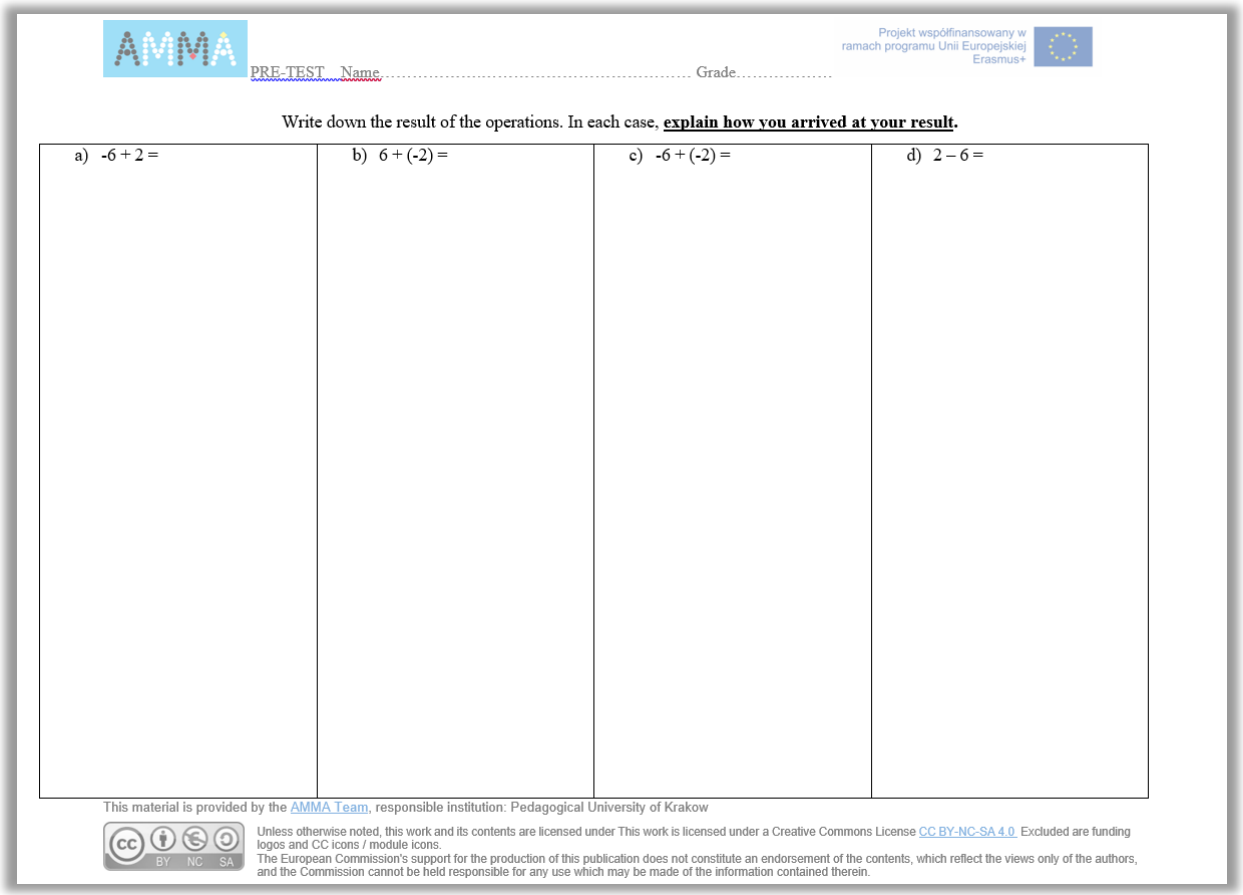
ACTIVITY 1: PRE-TEST

{Comment: The students do not refer to the tokens during the PRE-TEST as they simply are not familiar with them at this stage, although they are likely to use this interpretation in the POST-TEST - i.e. perhaps the effectiveness of tokens will be seen}

Individual work

[Appendix: A_ENG_Pre-test]

"Write down the result of the operations. In each case, explain how you arrived at your result."



AMMA

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PRE-TEST Name..... Grade.....

Write down the result of the operations. In each case, **explain how you arrived at your result.**

a) -6 + 2 =	b) 6 + (-2) =	c) -6 + (-2) =	d) 2 – 6 =

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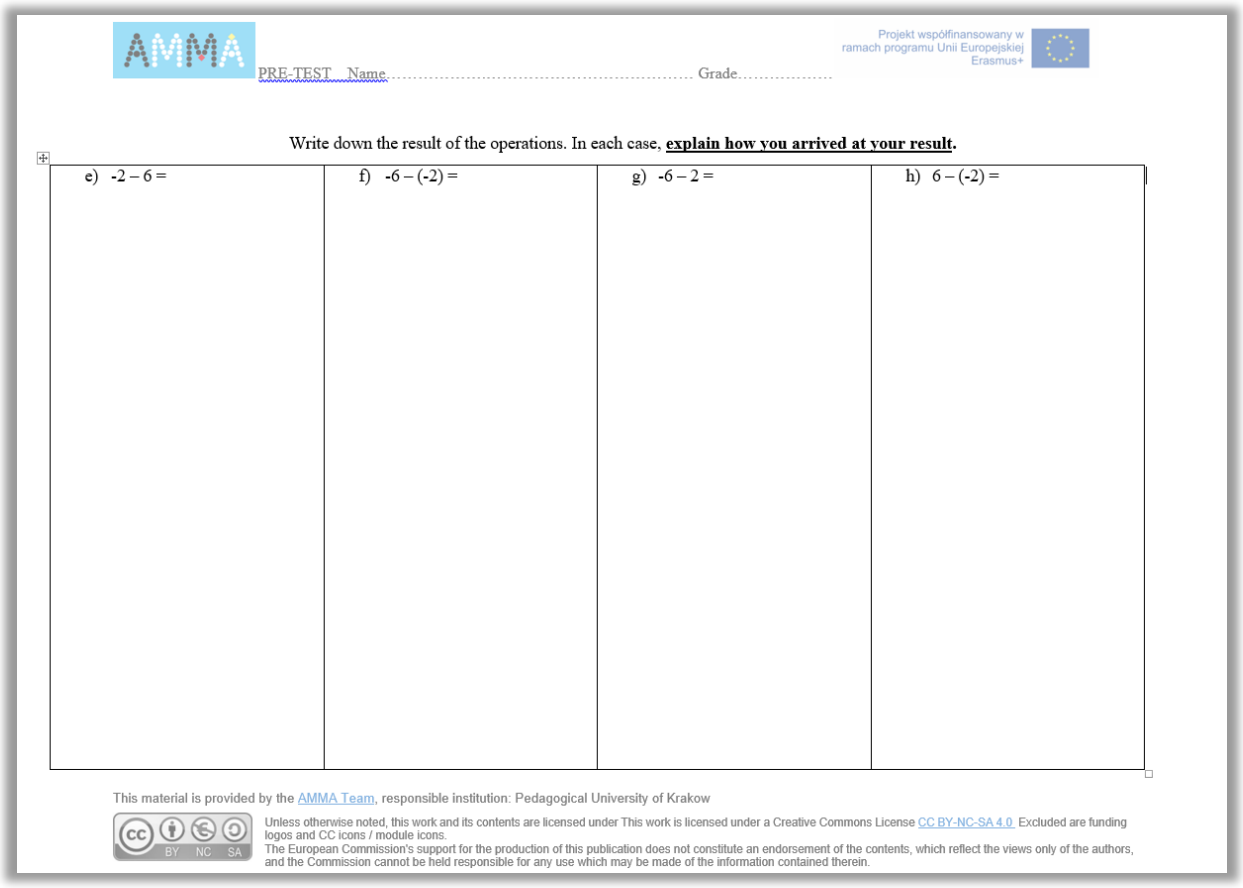


PRE-TEST Name..... Grade.....

Projekt współfinansowany w ramach programu Unii Europejskiej Erasmus+

Write down the result of the operations. In each case, **explain how you arrived at your result**.

e) -2 – 6 =	f) -6 – (-2) =	g) -6 – 2 =	h) 6 – (-2) =

ACTIVITY 2: Introduction to negative and integer numbers - formation of concepts

RULES FOR PLAYING WITH TOKENS

The students are given tokens: 10 tokens of each type per student.

The teacher has either:

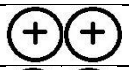
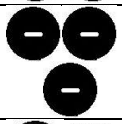
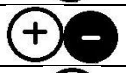
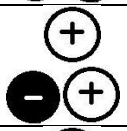
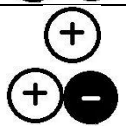
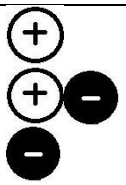
- *Ten large, magnetic tokens of each type to pin on the board (PL),*
- *Or an interactive whiteboard where the tokens can be placed and moved around (SK).*

We are introducing the agreement 	***Together with the students, we agree the on terminology***
White → 1 „positive" token → has a value of 1	Possible proposals from students: *- white and black form a neutral pair* *- white and black form a pair which has a value of 0* *- white and black form a pair which represents 0* *- white and black form a pair of opposing tokens* *- white and black eat each other* ***- white and black (two tokens of different colours) annihilate/disappear***
Black → 1 "negative" token → has a value of -1	
- Two tokens of different colours cancel each other out. ***-How could we phrase this differently?***	

ACTIVITY 3: READING THE VALUE OF A NUMBER FROM A MODEL

ACTIVITY 3a

The teacher places the tokens [magnetic or virtual] on the board. The students declare and justify the value represented by the tokens.

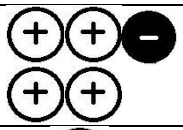
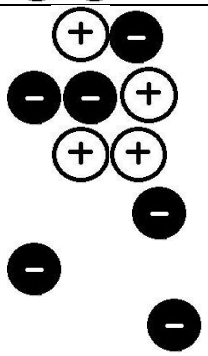
No.	Teacher places tokens on board	Students respond	*Remarks* / Issues for discussion
1.		1	
2.		2	
3.		-3	*REMARKS:* *- We arrange the tokens in a disorderly, "chaotic" manner* *- If students spontaneously start talking about addition, praise them and say that we will discuss it in a moment*
4.		0	
5.		1	
6.		1 REMARK: - *What do you notice?* *We note that the situation is the same as above*	
7.		0	

ACTIVITY 3b

The teacher tells the students which tokens to arrange (he/she can arrange them himself/herself at the same time). The students arrange them independently. They answer and justify the values represented by the tokens.

Approx. 4 examples are chosen by the teacher. The teacher or student then states how many tokens of a particular colour need to be taken (the table only presents the steps).

This is followed by class discussion about the most efficient way of placing the tokens.

Teacher says	Students arrange the tokens	Students supplement the value	*Remarks* / Issues for our discussion
take 4 white tokens and 1 black		3	*– Students place whatever they want on the desk; if some do it choatically, and others neatly, one below the other, then there is an opportunity to discuss these approaches – which is more useful* *- If ordering (stacking whites over blacks or vice versa) does not happen naturally and spontaneously for the students, deliberately ask questions about the most efficient way of placing the tokens.*
A few whites and blacks, for example, like this:		-2	

ACTIVITY 3c

Students work in pairs. Each student arranges his/her set of tokens in pairs. Their classmate reads out the numerical value of the given set.

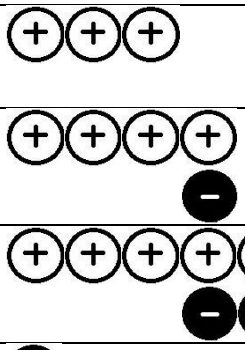
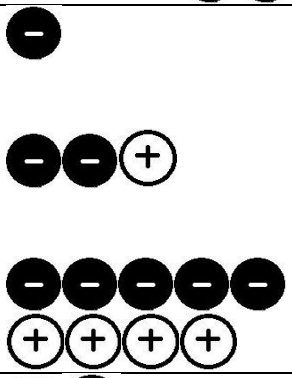
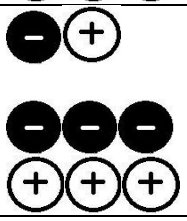
ACTIVITY 4: REPRESENTATION OF INTEGERS ON A MODEL

Students work individually.

Perform the task in the form of a "rare case" competition: display the given expression using tokens so that no one else the same representation as you. The rarest solution wins. (Scoring: everyone gets as many points as the number of students with that solution. The student with the fewest points wins.)

- Construct an interesting set of tokens with a set value (sequentially: 3, -1, 0)

{The table illustrates the expected sample results}.

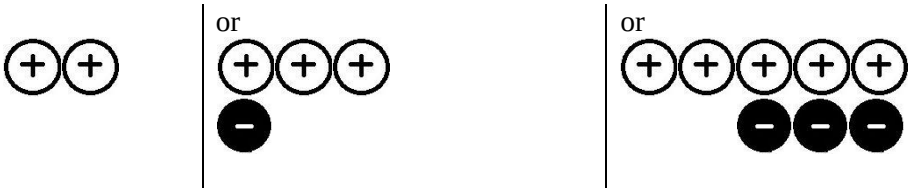
The given number	Possible models
3	
-1	
0	

- Let's draw some example layouts for the number 2 in the notebook.

- How should we draw it to make it easy to see what number it is? (Different tokens one below the other to make it easy to cross out in pairs)

Sample note:

Number 2:



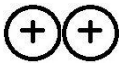
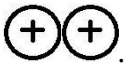
Remark:

Because of the time spent on drawing, an arrangement can be made with the students to draw the tokens with a + or – sign, without colouring them in. The signs in the circles should not be confused with the addition and subtraction signs.

Of course, you can leave the colour in, but there will be a lot of painting (more work and time than simply putting + and - in the circles).

INTRODUCTION TO ADDITION *{ from model to mathematical notation}*

Collaborative work - the teacher uses large magnetic or virtual tokens, working at the board, recording the activity simultaneously:

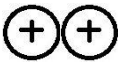
- *- What action will describe the following situations:*
  - *I have got*  *and also*  *. How much do I have in total?*
  - *I have got*  *and I put in*  *. How much do I have in total?*
  - *I have got*  *and I put in*  *. How much do I have in total?*

Students write down: 2+3 = 5 *and* 1+2 = 3 *and* (-2) + (-4) =(-6) *or* -2 + (-4) = -6

REMARK!

We agree that both notations are correct (-2) + (-4) = (-6) or -2 + (-4) = -6***. A parenthesis is necessary if two characters are next to each other.***

CONTRIBUTION: PUT IN, PLACE means ADD

- *- How many tokens do I have in total:*
  - *To*  *I add*  *.*
  - *Students answer: together I have two whites and write down* 3 + (-1) = 2
  - *To*  *I add* 

Students answer: together I have one black and write down (-3) + 2 = -1

{If the students have difficulties, another example can be done together.}

ACTIVITY 5. ADDITION EXERCISE

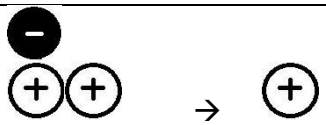
Collaborative working

- Now we are going to deal with various situations of addition, some of which are so trivial that they might seem funny.

- *E.g. 2+3 – how could this be represented by the tokens? {Students give a verbal description}*
- *And what would the action (-2) + (-3) look like on the tokens? {Students give a verbal description: I take 2 black tokens and I put in 3 more - in total I have 5 black tokens}*

Work in pairs

- That was trivial – time for something a little more difficult – something that is already presented with the tokens in pairs: one person describes the action -1+2, the other 2+(-1). Please justify the result on the tokens.

Task activity	**Students arrange the model and justify the result**	***Remarks***
A) -1+2 = *(one person in pair)* 2+(-1) = *(second person in pair)* *We do successive examples in a similar way, until the students notice the commutative property.*		REMARKS: - *We ask the students to speak. The language of justification is important: adding a black token to a white token will remove it while the white one remains, i.e. the value is 1.* - *Students should notice that the records of the two operations (commutative property) can be represented by the same arrangement of tokens, as it is a matter of counting in the correct order. If they don't notice, then an appropriate question should be asked, e.g. after the first four actions:* *What do you notice?* - *Note the brackets and the statement so that the operation sign is read as an action (not just "plus"):* *minus 3 ADD 2*
B) -3+2 = 2+(-3) =		
C) 1+(-2) = -2+1 =		
D) -2+3 = 3+(-2)=		

Follow these examples with a note.

- *Write the chosen operation in your notebook and justify by illustrating in the figure how the result is formed from the corresponding tokens.*

ACTIVITY 6 ADDITION EXERCISE - continuation

Individual work.

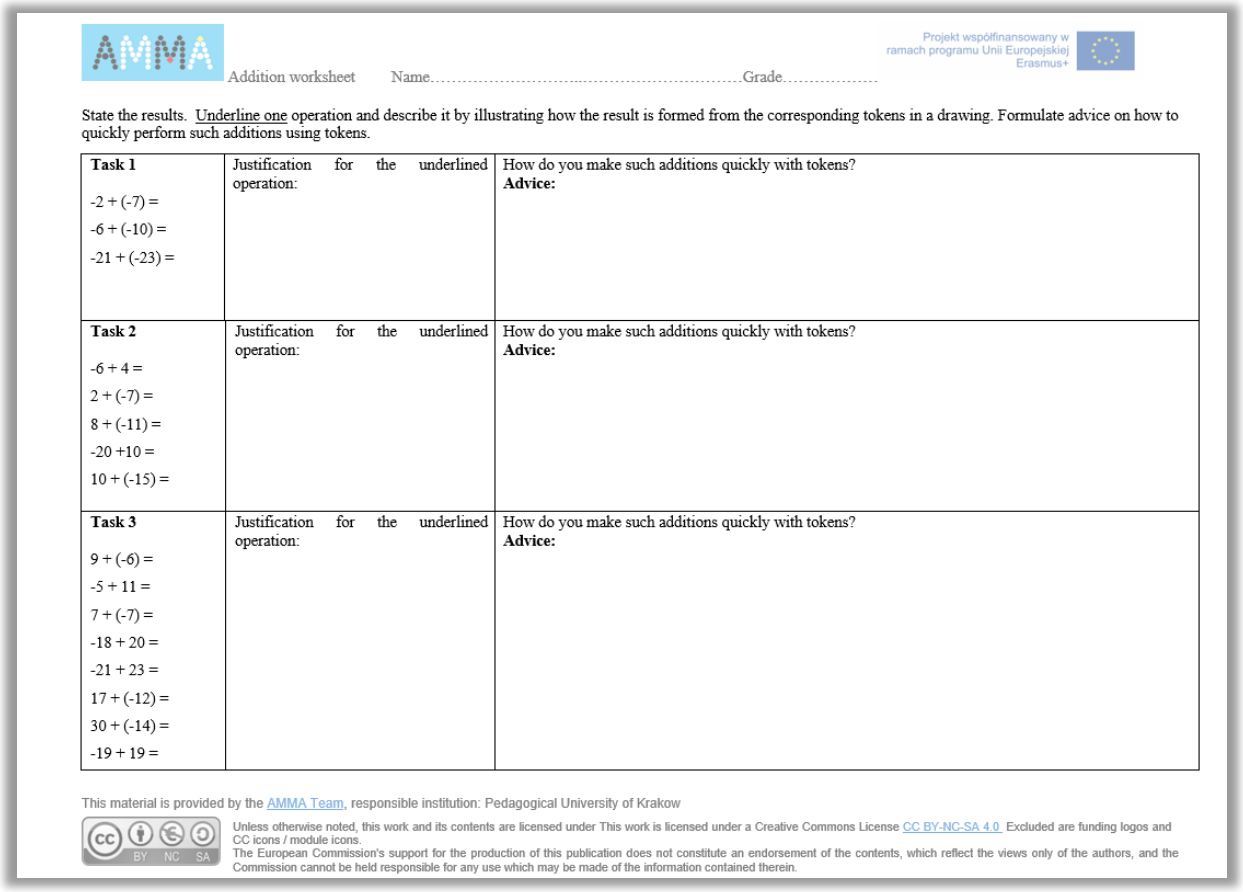
**) Indicates examples where there are not enough tokens for students. The students' work sheets do not include this notation.*

Contents of the worksheet:

State the results. Underline one operation and describe it by illustrating how the result is formed from the corresponding tokens in a drawing. Formulate advice on how to quickly perform such additions using tokens.

Task 1 -2 + (-7) = -6 + (-10)* = -21 + (-23)* =	Justification for the underlined operation:	How do you make such additions quickly with tokens? **Advice:**
Task 2 -6 + 4 = 2 + (-7) = 8 + (-11)* = -20 +10* = 10 + (-15)* =	Justification for the underlined operation:	How do you make such additions quickly with tokens? **Advice:**
Task 3 9 + (-6) = -5 + 11* = 7 + (-7) = -18 + 20* = -21 + 23* = 17 + (-12)* = 30 + (-14)* = -19 + 19* =	Justification for the underlined operation:	How do you make such additions quickly with tokens? **Advice:**

[Appendix: A_ENG_ Addition worksheet]



Addition worksheet Name……Grade……

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State the results. Underline one operation and describe it by illustrating how the result is formed from the corresponding tokens in a drawing. Formulate advice on how to quickly perform such additions using tokens.

Task 1 -2 + (-7) = -6 + (-10) = -21 + (-23) =	Justification for the underlined operation:	How do you make such additions quickly with tokens? **Advice:**
Task 2 -6 + 4 = 2 + (-7) = 8 + (-11) = -20 +10 = 10 + (-15) =	Justification for the underlined operation:	How do you make such additions quickly with tokens? **Advice:**
Task 3 9 + (-6) = -5 + 11 = 7 + (-7) = -18 + 20 = -21 + 23 = 17 + (-12) = 30 + (-14) = -19 + 19 =	Justification for the underlined operation:	How do you make such additions quickly with tokens? **Advice:**

ACTIVITY 7. ADDITION EXERCISE – for gifted students who finish their worksheet early

Calculate:

a) -1 + (-3) + (-8) =

b) -6 + (-7) + 5 =

c) -8 + 4 + (-7) + 14 =

d) -23 + (-17) + (-23) =

e) -56 + (-31) + 26 =

f) -11 + 7 + (-29) + 23 =