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Teacher Guide
Description of the lesson series

Title	**Algebraic expressions and their addition and subtraction using tokens**
Time	*5-8 school hours (depending on the students' pace and learning level)*
Grade	*Grades 6-8 (students 12-15 years old) or Grade 9 (for students with difficulties in learning mathematics)*
Aim of the lesson cycle and its brief description	*The aim of this series of lessons is to shape the concept of an algebraic expression and its opposition, as well as the addition and subtraction of such expressions using tokens.* *The scenario can be used both in younger grades as an introduction to algebraic expressions and for repetition lessons with students in older grades.* *As students play with the concrete model (tokens), they build up the concept of the algebraic expression and its opposition, and develop an understanding of the operation of addition as adding tokens, and subtraction as taking away tokens.* *Through this, students undertake mathematical modelling.*
Teaching materials	*Each student is given 10 tokens of each colour (white/black) and each shape: (round/oblong/square), for a total set of 60 tokens, to use as tools during the lessons.*

A linguistic note on working with tokens in the context of integers and algebraic expressions:

In our scenarios, we are careful to keep the two worlds - the world of mathematics, i.e. abstractions, and the world of real objects - in our case tokens - linguistically separate. Thus, in the context of tokens, we use terms that describe their appearance: white/black round/ oblong/square token rather than the short-form white circle/rectangle/square. Similarly, in the context of tokens, we mention placing and taking away tokens – while in the context of mathematics, we discuss addition and subtraction operations. We also make a point of verbally reading action signs as add/subtract, rather than just naming them plus/minus signs. We believe that modelling arithmetic and algebraic expressions with clarity and linguistic correctness in mind is of great value and is highly recommended.

PART 4

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Part 4

Topic: Addition and subtraction of complex algebraic expressions using tokens

ACTIVITY 1: Addition of complex algebraic expressions

- In mathematics, we often add and subtract more complex algebraic expressions using brackets. In the next part of the lesson, we will learn how to use them.

Example 1)

The teacher arranges two algebraic expressions on the board using tokens - on the left side, he places 2 white square tokens, and on the right side, he places 3 white oblong ones and 1 black round one. Teacher then asks the students what expressions have been presented; they write down both groups of tokens with their algebraic expressions (as shown below).

- What algebraic expression can we recognize?

U: $2x^2$, $3x + (-1)$,

- Can this expression be written in a shorter form?

{We want to see the notation $3x - 1$ Let's write it down this way.

- We now want to add the ***whole algebraic expression*** $3x - 1$ *to the expression* $2x^2$*. That is, we want to find the expression that results from the addition of:* $2x^2 + (3x - 1)$. {Write down the sum of these expressions and highlight the addition sign of the sum in colour}

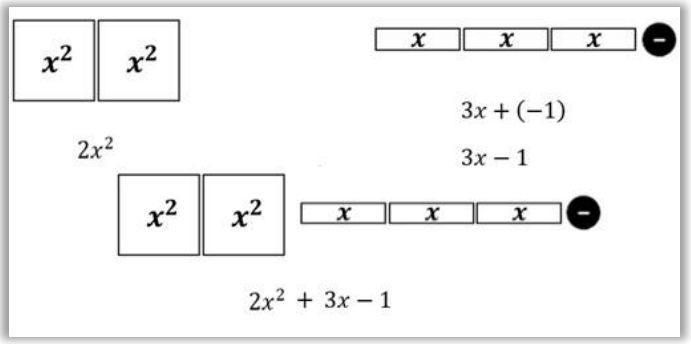
- What would we do if we wanted to get one expression which is the sum of the two?

{We are aiming for the students to say that the tokens should be joined together so that one expression is formed}.

- What is the sum of these two expressions?

{If the answer is: $2x^2 + 3x + (-1)$ then we ask: Can we shorten this - without the parenthesis? Answer: Without the brackets it is $2x^2 + 3x - 1$. Slide the tokens into the middle of the board.

- *Students write down the contents of the board to their notebooks:*



$$2x^2 + (3x + (-1)) = 2x^2 + (3x - 1) = 2x^2 + 3x - 1$$

The teacher emphasises the equality of the algebraic expressions written in different forms.

Example 2)

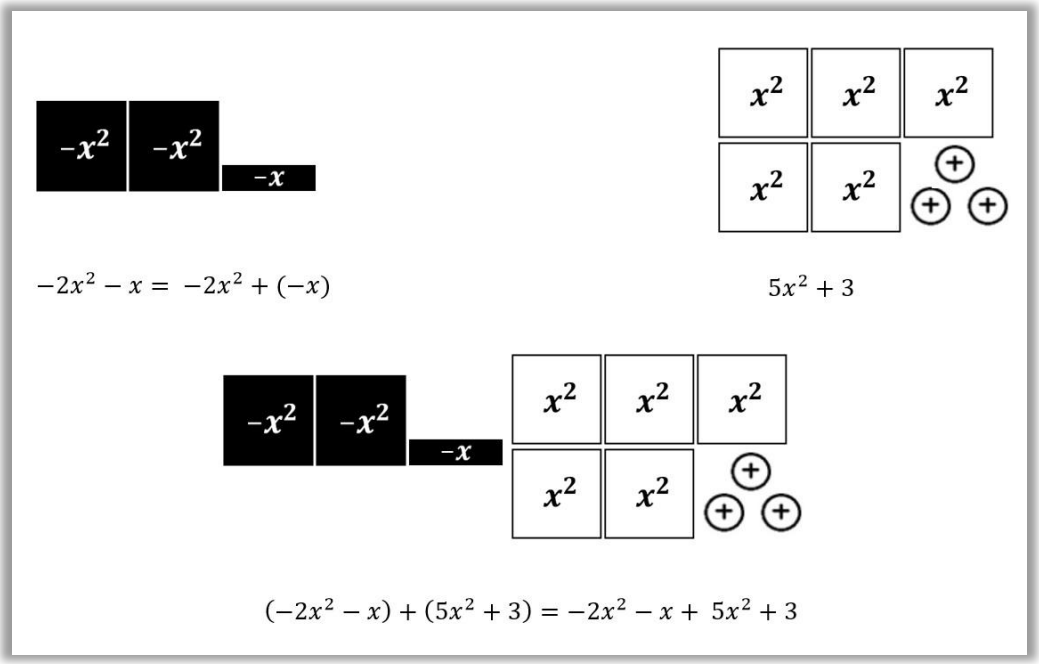
The teacher writes down the addition of two algebraic expressions on the board: $(-2x^2 - x) + (5x^2 + 3)$.

- *Now, we would like to add the following two algebraic sums together → A notation arises:* $(-2x^2 - x) + (5x^2 + 3)$.

- *Let's illustrate this by using tokens*

  {We represent the expressions with tokens: $-2x^2 - x$ and $5x^2 + 3$, then slide the tokens together and read the expression $-2x^2 - x + 5x^2 + 3$}

The students are to write down the example in their notebooks, without modification:



- *Would you now be able to write this algebraic expression more simply?*

  S: Yes: $(-2x^2 - x) + (5x^2 + 3) = -2x^2 - x + 5x^2 + 3$

Example 3)

The teacher writes the addition of two algebraic sums on the board: $x^2 + (-2x - 3)$.

- *Now we would like to add such two algebraic sums together→ The resulting notation:*

$$x^2 + (-2x - 3)$$

- Let's illustrate this with tokens

{We create these expressions with tokens. We use that: $-2x - 3 = -2x + (-3)$.

We have together: $x^2 + (-2x) + (-3)$.

- How can we write the result differently?

$$x^2 - 2x - 3$$

We notice that
$\boldsymbol{x^2 + (-2x - 3) = x^2 - 2x - 3.}$

Individual work

- *Carry out the activities by using tokens and record the results in your notebook.*

- $(4x^2 - 3) + x^2 =$
- $(3x + 4x^2) + (x - 5x^2) =$

ACTIVITY 2: Introduction of an expression opposite to complex algebraic expressions

Let's add these two complex algebraic expressions and calculate what this will amount to (with tokens, and record the result):

$(3x^2 + x) + (-3x^2 - x) = 3x^2 + x + (-3x^2) - x = 0$

What shall we call these two complex algebraic expressions: $3x^2 + x$ *i* $-3x^2 - x$*?*

OPPOSITE EXPRESSIONS

- *Provide two complex, opposite algebraic expressions and justify using tokens that it was indeed a good choice {students add the given expressions, they should return 0}*

Individual work (students do the exercise)

Create expressions opposite to:

- $-5x + 3$
- $9x^2 - 8x$
- $4x^2 + 2 - 6$
- $-3x - 7$

ACTIVITY 3: **Subtraction of complex algebraic expressions**

Example 1)

The teacher arranges two algebraic expressions on the board by using tokens - on the left side, he places 3 white square tokens, and on the right side, 1 white oblong and 2 black round ones. He then asks the students about the expressions that are presented.

- *What two expressions are shown on the board?*
  S: left: $3x^2$, *right* $x - 2$

- *And how would we write them down, using an operation, if we wanted to subtract the expression on the left from the expression on the right?*
  S: $3x^2 - (x - 2)$
  {we write down the difference of these expressions and highlight the subtraction sign between them in colour }.

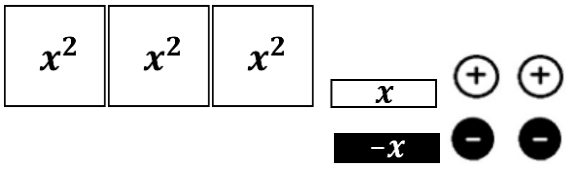
- *And what would we do if we wanted to get the result of this operation?*

  {2 ways - emphasis on adding the opposite expression}

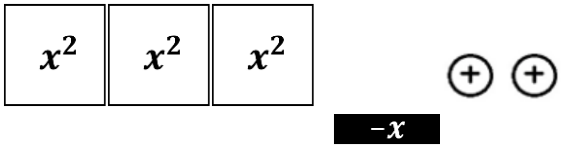
The first way – taking away

Add 1 neutral pair of oblong tokens and 2 neutral pairs of round tokens to the 3 white square tokens.

To take away 1 white oblong token and 2 black round tokens, we must first have them, so we add 1 neutral pair of oblong tokens and 2 neutral pairs of round tokens to the 3 white square tokens.



These tokens are then taken away: 1 white oblong and 2 black round ones.



Tokens left: 3 white squares, 1 black oblong, and 2 white rounds.

So $3x^2 - (x - 2) = 3x^2 - x + 2$

The second way - adding the opposite expression

That is, $3x^2 - (x - 2) = 3x^2 + (-x + 2)$**.**

Example discussion:

- *What does it mean to subtract an expression?*
  S: i.e. to add the opposite expression to the one being subtracted, i.e. $3x^2 - (x - 2) = 3x^2 + (-x + 2)$**.**
- *What (without using brackets) is the sum of these two expressions?*
  S: $3x^2 - x + 2$

The students write down the suggested methods in their notebooks.

Example 2)

The teacher writes this expression on the board: $(4 - 2x) - (3x - 2) =$.

- *What do we have here?*
  S: We have the subtractions of two algebraic expressions

- *How do I simplify this with tokens?*

  S: Subtract means to add an expression opposite to the one being subtracted i.e. I add tokens to the first expression: 3 black oblong ones and 2 white round ones:

  $(4 - 2x) - (3x - 2) = 4 - 2x + (-3x + 2) = 4 - 2x - 3x + 2$

  {The emphasis is on adding the expression opposite to the one subtracted - we no longer do it two ways}

Students write this down in their notebooks.

Example 3)

The teacher writes this expression on the board: $2x - (-4x^2 + 1) =$.

{note that to subtract means to add the opposite expression to the one being subtracted i.e. I add the opposite expression to $-4x^2 + 1$, so by using tokens: 4 white square ones and 1 black round one: $2x - (-4x^2 + 1) = 2x + (4x^2 - 1) = 2x + 4x^2 - 1$}

Individual work

- *Carry out these operations by using tokens and record the results in your notebook.*

- $6 - (-5x^2 + 4) =$
- $(4x^2 - 7x) - (2x^2 + 3x) =$

We check the results together.

ACTIVITY 4: Game

Game - cards:

Description: 2- , 3- or 4-player game (at the teacher's discretion). The instructions below are for a game between four people.

There are 4 sets of cards in different colours. Each person receives 1 set of cards consisting of 4 cards with operations and one additional card with the answers to the set.

Person no. 1 draws an activity card from person no. 2, person no. 2 from person no. 3, person no. 3 from person no. 4 and person no. 4 from person no. 1. The persons simultaneously solve the drawn activity on the card. Each person gets 1 point for a correct solution. In addition, the first person to correctly solve the activity gets a bonus point. The game continues as such until the cards are exhausted. After each round, the points are recorded in a table:

N	Person No. 1	Person 2	Person 3	Person 4
Round 1				
Round 2				
Round 3				
Round 4				

The person with the highest number of points wins.

Sets:

Person No. 1 (blue)

N	**Action**	**Answer**
1.	$4x^2 + (x - 8x^2) =$	$-4x^2 + x$
2.	$(2x - 5) + (6x - 1) =$	$8x - 6$
3.	$3x - (x - 7) =$	$2x + 7$
4.	$(7 + x^2) - (-3x^2 - 1) =$	$4x^2 + 8$

Person No. 2 (red)

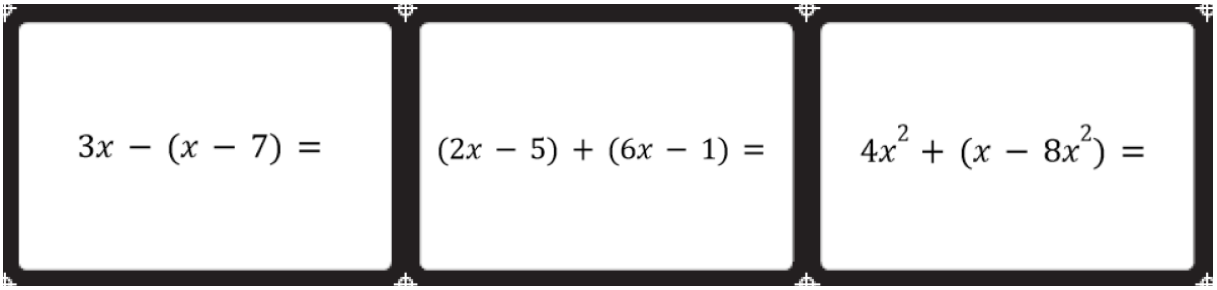
N	Action	Answer
1.	$(2x - 5x^2) + 3x^2 =$	$-2x^2 + 2x$
2.	$(-x - 3) + (4 + 6x) =$	$5x + 1$
3.	$7x - (2x - 8) =$	$5x + 8$
4.	$(2x^2 - 6) - (4 + 4x^2) =$	$-2x^2 - 10$

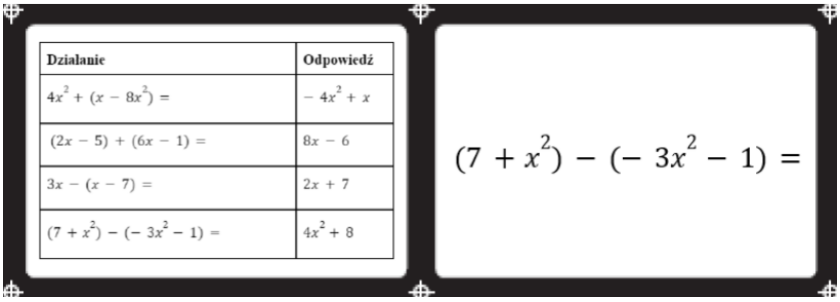
Person No. 3 (green)

N	Action	Answer
1.	$2x^2 + (3x - 5x^2) =$	$-3x^2 + 3x$
2.	$(-6 + 4x) + (-6x + 4) =$	$-2x - 2$
3.	$(10x - 10) - 2x =$	$8x - 10$
4.	$(9x^2 + 3) - (x^2 - 2) =$	$8x^2 + 5$

Person No. 4 (yellow)

N	Action	Answer
1.	$(4x - 8x^2) + 5x^2 =$	$-3x^2 + 4x$
2.	$(-2x - 3) + (6 + 6x) =$	$4x + 3$
3.	$(2x - 9) - x =$	$x - 9$
4.	$(4x^2 + 3) - (2 - x^2) =$	$5x^2 + 1$





ACTIVITY 5: COMPLEX ALGEBRAIC ADDITION AND SUBTRACTION – EXERCISES (continuation)

Task 1

a) $6x^2 + (6x - 8x^2) =$

b) $(4x - 2) + (5x - 3) =$

c) $(35 + 10x^2) + (21x^2 - 3) =$

Task 2

a) $x - (14x - 22) =$

b) $(3x^2 + 1) - (2x^2 - 1) =$

Task 3

a) $(x + x^2) + (x - 4) - (5 - x^2) =$

b) $(x^2 + 5x - 6) + (2x^2 - x + 8) =$

[Appendix: Test_Algebraic expressions]

Name ……..... class……

TEST - ALGEBRAIC EXPRESSIONS

Task 1
Draw the models of the given algebraic expressions by using tokens.

a) $-2x + (-x^2) + 3$	**b)** $-x + 2x + x^2 + (-2)$

Task 2
Simplify the expressions. Explain how to perform the following operations. Illustrate this with token drawings.

a) $-6x + (-2x) =$ Explanation:	b) $-3x^2 + 5x^2 =$ Explanation:

Task 3

Simplify the expressions. Explain how to perform the following operations. Illustrate this with token drawings.

a) $-4x^2 - (-4x^2) =$ Explanation:	b) $3x - 7x =$ Explanation:

Task 4

The expression opposite to $\mathbf{-4x^2 + 5}$ is: ……..

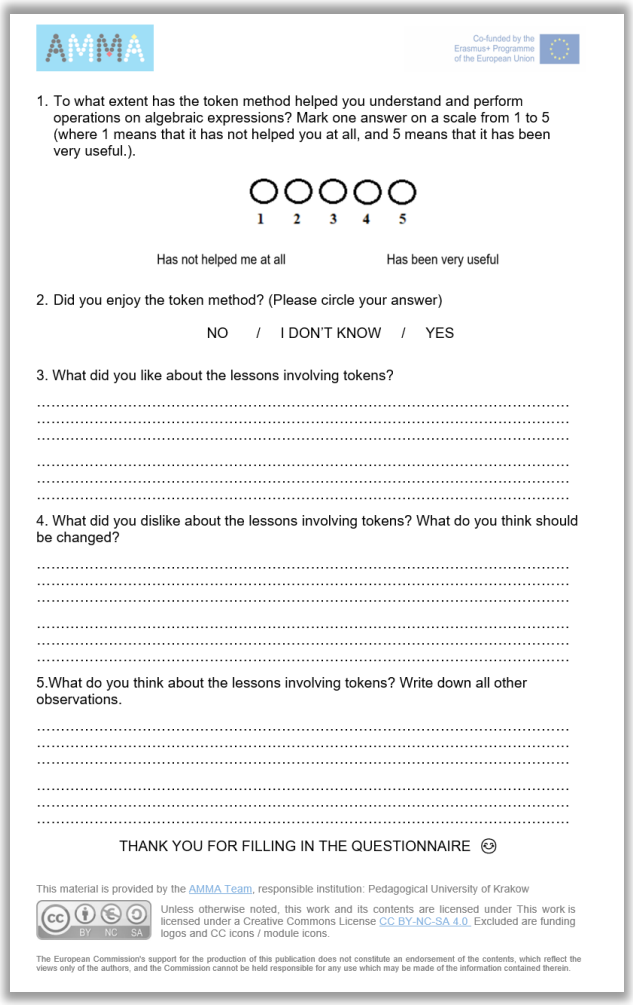
Present a model of the expression opposite to: $\mathbf{-4x^2 + 5}$ by using tokens:

Task 5

Simplify the expressions. Justify your result by making appropriate drawings involving tokens.

a) $\mathbf{1 - (-4x^2 + 5)} =$	b) $\mathbf{-3x^2 + (-4x) - x + 3 + x^2} =$

[Appendix: B_ENG_Questionnaire Final]



1. To what extent has the token method helped you understand and perform operations on algebraic expressions? Mark one answer on a scale from 1 to 5 (where 1 means that it has not helped you at all, and 5 means that it has been very useful.).

○ ○ ○ ○ ○
1 2 3 4 5

Has not helped me at all — Has been very useful

2. Did you enjoy the token method? (Please circle your answer)

NO / I DON'T KNOW / YES

3. What did you like about the lessons involving tokens?

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4. What did you dislike about the lessons involving tokens? What do you think should be changed?

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5.What do you think about the lessons involving tokens? Write down all other observations.

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THANK YOU FOR FILLING IN THE QUESTIONNAIRE ☺

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QUESTIONNAIRE – ALGEBRAIC EXPRESSIONS

1. To what extent has the token method helped you understand and perform operations on algebraic expressions? Mark one answer on a scale from 1 to 5 (where 1 means that it has not helped you at all, and 5 means that it has been very useful.).

○ ○ ○ ○ ○
1 2 3 4 5

Has not helped me at all **It is very useful**

2. Did you enjoy the token method? (please circle your answer)

NO / I DO NOT KNOW / YES

3. What did you like about the lessons involving tokens?

4. What did you dislike about the lessons involving tokens? What do you think should be changed?

5.What do you think about the lessons involving tokens? Write down all other observations.